Site percolation on sets in a metric space

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Let (X,d) be a metric space.
Let $\mathcal{P}$ be a countable collection of closed subsets of X of finite diameter.

Graph structure on $\mathcal{P}$: $S_1, S_2 \in \mathcal{P}$ adjacent if $S_1 \cap S_2 \neq \phi$.

Assume: (few large sets nearby)
There exist $C_1, C_2 \geq 0$ such that for all $S \in \mathcal{P}$, $\rho, t > 0$,

$$|\{S' \in \mathcal{P} : d(S,S') \leq \rho,\ \mathrm{diam}(S') \geq t\}| \leq e^{C_1 + C_2 \frac{\rho + \mathrm{diam}(S)}{t}}$$

Example: Packing of shapes in $\mathbb{R}^n$ whose volume is proportional to the n'th power of their diameter with a uniform proportionality constant.

Blue sets have diameter at least t.

E.g.: Circle packings, cube packings.

Theorem: There exists $p_0 = p_0(C_1, C_2)$ $= e^{-\alpha(C_1+C_2+1)}$ for some large $\alpha>0$ such that

$$\mathbb{P}_p(\exists \text{ connected comp. of open sets of infinite diameter}) = 0, \ \forall p < p_0.$$

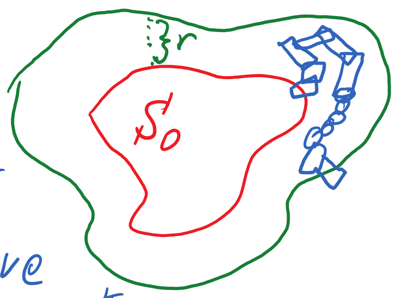
$\uparrow$ p-site percolation on the sets in $\mathcal{P}$. Open set = retained set.

Main lemma: There exists p_0 s.t. for all $p < p_0$, $S_0 \in \mathcal{P}$, $r \geq 0$, $k \in \mathbb{Z}$,

$$\mathbb{P}_p(S_0 \xrightarrow{\leq 2^k} r) \leq e^{C_2 \frac{\mathrm{diam}(S_0)}{2^{k-2}}} p^{\frac{3}{4}\frac{r}{2^{k+4}}}$$

if $r \geq \mathrm{diam}(S_0) \rightarrow \ \leq p^{\frac{r}{2^{k+5}}}$.

The event $\{S_0 \xrightarrow{\leq 2^k} r\}$:



Both S_0 and the blue sets are open.
Blue sets have diameter $\leq 2^k$.

Proof of main lemma:

... inf diam(S) > 0

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It suffices to prove when $\inf_{S\in\mathcal{P}} \operatorname{diam}(S)>0$
since every finite path has an S with minimal positive diameter.

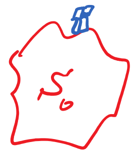
Scaling the metric we may and will assume that $\inf_{S\in\mathcal{P}} \operatorname{diam}(S)=1$.

We prove the lemma by induction on K.

Base case $K=0$: For $\{S_0 \xrightarrow{\le 1} r\}$ all used sets must have diameter exactly 1.

Number of paths of length L using sets of diam. 1, starting at S_0 and going to distance r is

$$\le e^{c_1+c_2 \operatorname{diam}(S_0)}\cdot\left(e^{c_1+c_2}\right)^{L-2}$$



prob. to be open for each path $= p^L$

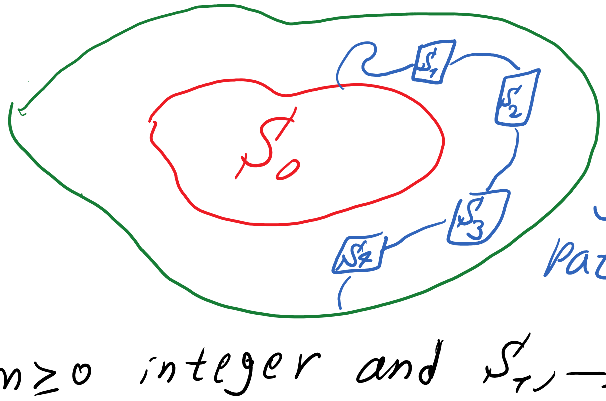
Necessarily have $L>\lceil r\rceil$.

$$\sum_{L=\lceil r\rceil+1}^{\infty} e^{c_1(L-1)+c_2 \operatorname{diam}(S_0)+(L-2)c_2}\cdot p^L \le$$

$$\le e^{4c_2 \operatorname{diam}(S_0)}\cdot p^{\frac{3}{4}\frac{r}{2^4}}$$ as required.

Induction step ($K\ge 1$): Assume lemma holds up to $K-1$ and prove for K.

Decompose the event $\{S_0 \xrightarrow{\le 2^K} r\}$ according to the sets with diam. in $[2^{K-1},2^K)$ that are used on the open path.



Blue S_i have diam. $\in[2^{K-1},2^K)$ and all other sets on the open path have diam. $<2^{K-1}$.

For $m\ge 0$ integer and $S_1,\dots,S_m\in\mathcal{P}$ with $d(S,S_i)<r$ and $\operatorname{diam}(S_i)\in[2^{K-1},2^K)$ $\forall i$

For $m \geq 0$ integer and $S_1, \ldots, S_m \subset$

with $d(S, S_i) < r$ and $\operatorname{diam}(S_i) \in [2^{k-1}, 2^k)$ $\forall i$

let $E_{S_0, S_1, \ldots, S_m} = \{ S_0 \xrightarrow{\leq 2^{k-1}} \text{neigh. of } S_1,\ S_1 \xrightarrow{\leq 2^{k-1}} \text{neigh. of } S_2,\ \ldots,\ S_m \xrightarrow{\leq 2^{k-1}} \text{dist. } r \text{ from } S_0 \text{ with all connections disjoint} \}$

Claim: $\mathbb{P}(E_{S_0, S_1, \ldots, S_m}) \leq \prod_{i=0}^{m-1} \min\{p, p^{\frac{d(S_i, S_{i+1})}{2^{k+4}}}\} \cdot \min\{p, p^{\frac{d(S_m, \text{dist. } r \text{ from } S_0)}{2^{k+4}}}\}.$

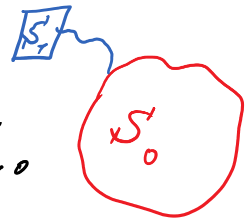
This is proved for $m \geq 1$. The case $m = 0$ is handled separately

proof: By the Van den-Berg–Kesten ineq.

$$\mathbb{P}(E_{S_0, S_1, \ldots, S_m}) \leq \prod_{i=0}^{m-1} \mathbb{P}(S_i \xrightarrow{\leq 2^{k-1}} \text{neigh. of } S_{i+1}) \cdot \mathbb{P}(S_m \xrightarrow{\leq 2^{k-1}} \text{dist. } r \text{ from } S_0)$$

What to do with $\mathbb{P}(S_0 \xrightarrow{\leq 2^{k-1}} \text{neigh. of } S_1)$?

Trick: It equals $\mathbb{P}(S_1 \xrightarrow{\leq 2^{k-1}} \text{neigh. of } S_0)$, by swapping the states of S_0 and S_1.



By the induction hypothesis,

$$\mathbb{P}(S_1 \xrightarrow{\leq 2^{k-1}} \text{neigh. of } S_0) \leq \min\{p, e^{8C_2} p^{\frac{3}{4}\frac{d(S_0, S_1)}{2^{k+3}}}\} \leq$$

since S_1 needs to be open

Assume $p \leq e^{-\alpha(C_1 + C_2 + 1)}$

$$\leq \min\{p, p^{\frac{d(S_0, S_1)}{2^{k+4}}}\}.$$

Apply similar reasoning (without swapping trick) to other terms. $\square$

Main lemma follows from claim by summing over all m and $S_1, \ldots, S_m$.